Deduction modulo and truth values algebras

(a uniform approach to models and cut elimination)

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I. Deduction modulo

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Prelude : How do you know what you know ?

How do you know that $13 \times 17 = 221$?

How do you know that $p \neq 0 \Rightarrow (n^2 \neq 2p^2)$?

How do you know that $x * (y^{-1} * y) = x$ in all groups ?

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Prelude : How do you know what you know ?

How do you know that $13 \times 17 = 221$?

How do you know that $p \neq 0 \Rightarrow (n^2 \neq 2p^2)$?

How do you know that $x * (y^{-1} * y) = x$ in all groups ?

Two ways to solve problems : compute normal forms, find a proof

Use these two ideas jointly

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Frege-Hilbert style proof

Theory : (decidable) set of formulae called axioms

Set of deduction rules : (decidable) set of tuples of formulae

$$\frac{A_1 \ldots A_n}{A_{n+1}}$$

Proofs : trees labeled by formulae (leaves : axioms, internal nodes : rules)

$$\frac{\dfrac{A_1 \ldots A_n}{B_1} \quad B_2 \;\ldots\; \dfrac{\ldots}{B_p}}{C}$$

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Natural deduction

To prove $A \Rightarrow B$: assume A (new axiom / hypothesis) and prove B

Deduction rules transform the conclusion and also the axioms

Nodes labeled by pairs $\Gamma \vdash A$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

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Introduction and elimination rules

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-elim}$$

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Cuts

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B} \wedge\text{-intro}}{\Gamma \vdash A} \wedge\text{-elim}$$

Can be eliminated : π_1

A positive definition of cut free proofs

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A proof is cf if either

– it ends with the axiom rule

– it ends with an introduction rule and its sub-proofs are cf

– it ends with an elimination rule, its sub-proofs are cf and and its major sub-proof does not end with an introduction rule (neutral)

Why do we care ?

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Cut free proofs are easier to discover (no guessing of $A \wedge B$) : completeness of proof search methods

(Constructive) cut free proofs (with no axioms) end with an introduction rule

$$\frac{(\Gamma) \vdash (t/x)A}{(\Gamma) \vdash \exists x\ A}\ \exists\text{-intro}$$

Witness property (proofs as a programming language)

etc.

Axioms mess up cut free proofs

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(Constructive) cut free proofs need not end with an introduction rule *e.g.*

$$\frac{}{\exists x\ A \vdash \exists x\ A}\ \text{Axiom}$$

Deduction modulo

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Proof : sequence of deduction steps

Theory : set of axioms

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Deduction modulo

Proof : sequence of deduction steps and computation steps

Theory : set of axioms and computation rules

A terminating and confluent system of computation rules

Computation rules apply to terms, *e.g.*

$$0 + y \longrightarrow y$$

and to atomic formulae, *e.g.*

$$x \subseteq y \longrightarrow \forall z \, (z \in x \Rightarrow z \in y)$$

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Congruence

The computation rules define a congruence on formulae *e.g.*

$$(2 \times 2 = 4) \equiv (4 = 4)$$

Smallest relation that

– is an equivalence relation

– is a congruence (compatible with all the symbols)

– contains $\theta l \equiv \theta r$ for each computation rule $l \longrightarrow r$

The congruence $\equiv$ is decidable (thanks to termination and confluence)

Deduction rules

Deduction rules parametrized by the congruence $\equiv$, *e.g.*

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-elim}$$

$$\frac{\Gamma \vdash C \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-elim if } C \equiv (A \Rightarrow B)$$

Deduction modulo explained to your grand mother
(proofs and certificates)

Need to prove 221 is composite but too lazy to prove it myself

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Deduction modulo explained to your grand mother
(proofs and certificates)

Need to prove 221 is composite but too lazy to prove it myself

- as you can check yourself
- because 13 is a divisor
- because $221 = 13 \times 17$
- because

```
      2̸
      1 7
      1 3
    -----
      5 1
   ¹1 7
    -----
    2 2 1
```

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- *C* defined by computation rules

$$\frac{}{\Gamma \vdash_{\mathcal{R}} C(221)}\ \top\text{-intro}$$

(as you can check yourself)

- $|$ defined by computation rules $C(x) = \exists y\ y \mid x$

$$\frac{\dfrac{}{\Gamma \vdash_{\mathcal{R}} 13 \mid 221}\ \top\text{-intro}}{\Gamma \vdash_{\mathcal{R}} \exists y\ (y \mid 221)}\ \exists\text{-intro}$$

(because 13 is a divisor)

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– $\times$ defined by computation rules *C*$(x) = \exists y \exists z \; x = y \times z$

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash_{\mathcal{R}} \forall x \; (x = x)}{\Gamma \vdash_{\mathcal{R}} 221 = 13 \times 17}\ \forall\text{-elim}}{\Gamma \vdash_{\mathcal{R}} \exists z \; (221 = 13 \times z)}\ \exists\text{-intro}}{\Gamma \vdash_{\mathcal{R}} \exists y \exists z \; (221 = y \times z)}\ \exists\text{-intro}$$

(because $221 = 13 \times 17$)

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– only axioms *C*$(x) = \exists y \exists z \; x = y \times z$

$$\cfrac{\cfrac{\cfrac{\ldots}{\Gamma \vdash_{\mathcal{R}} 221 = 13 \times 17}}{\Gamma \vdash_{\mathcal{R}} \exists z \; (221 = 13 \times z)}\ \exists\text{-intro}}{\Gamma \vdash_{\mathcal{R}} \exists y \exists z \; (221 = y \times z)}\ \exists\text{-intro}$$

(every step of the multiplication)

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Business as usual (the equivalence lemma)

For each congruence $\equiv$, there is a set of axioms $\mathcal{T}$ such that

$$\Gamma \vdash_{\equiv} A$$

iff

$$\mathcal{T}, \Gamma \vdash A$$

e.g. $\mathcal{T} = \{\overline{\forall}\,(P \Leftrightarrow Q) \mid P \equiv Q\}$

Nothing new from the provability point of view

Something new from the proof structure point of view

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Purely computational theories

No axioms

Only computation rules and deduction rules

Examples : simple type theory, set theory, arithmetic

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Cut elimination

Cut elimination may be lost in deduction modulo *e.g.*

$P \longrightarrow (P \Rightarrow Q)$

$$
\dfrac{\dfrac{\dfrac{\overline{P \vdash P \Rightarrow Q} \qquad \overline{P \vdash P}}{P \vdash Q}}{\vdash P \Rightarrow Q}\ \Rightarrow\text{-intro} \qquad \dfrac{\dfrac{\overline{P \vdash P \Rightarrow Q} \qquad \overline{P \vdash P}}{P \vdash Q}}{\vdash P}}{\vdash Q}\ \Rightarrow\text{-elim}
$$

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Cut elimination

But when we have cut elimination for a purely computational theory

Cut free proofs are easier to discover : completeness of proof search methods

(Constructive) cut free proofs end with an introduction rule

Witness property

etc.

Axioms mess up cut free proofs, but rewrite rules do not

A full example : arithmetic

$$\forall x\ (x = x)$$
$$\forall x\ \forall y\ (x = y \Rightarrow (x/z)A \Rightarrow (y/z)A)$$
$$\forall x \forall y\ (S(x) = S(y) \Rightarrow x = y)$$
$$\forall x\ \neg 0 = S(x)$$
$$(0/z)A \Rightarrow \forall x\ ((x/z)A \Rightarrow (S(x)/z)A) \Rightarrow \forall n\ (n/z)A$$
$$\forall y\ 0 + y = y$$
$$\forall x \forall y\ S(x) + y = S(x + y)$$
$$\forall y\ 0 \times y = 0$$
$$\forall x \forall y\ S(x) \times y = x \times y + y$$

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A full example : arithmetic

$$\forall x \, (x = x)$$
$$\forall x \, \forall y \, (x = y \Rightarrow (x/z)A \Rightarrow (y/z)A)$$
$$\forall x \forall y \, (S(x) = S(y) \Rightarrow x = y)$$
$$\forall x \, \neg 0 = S(x)$$
$$(0/z)A \Rightarrow \forall x \, ((x/z)A \Rightarrow (S(x)/z)A) \Rightarrow \forall n \, (n/z)A$$
$$0 + y \longrightarrow y$$
$$S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0$$
$$S(x) \times y \longrightarrow x \times y + y$$

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A full example : arithmetic

$$\forall x \, (x = x)$$
$$\forall x \, \forall y \, (x = y \Rightarrow (x/z)A \Rightarrow (y/z)A)$$
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A full example : arithmetic

$$\forall x\ (x = x)$$

$$\forall x\ \forall y\ (x = y \Rightarrow (x/z)A \Rightarrow (y/z)A)$$

$$\forall x \forall y\ (S(x) = S(y) \Rightarrow x = y)$$

$$\forall x\ \neg 0 = S(x)$$

$$(0/z)A \Rightarrow \forall x\ ((x/z)A \Rightarrow (S(x)/z)A) \Rightarrow \forall n\ (n/z)A$$

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0$$

$$S(x) \times y \longrightarrow x \times y + y$$

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A full example : arithmetic

$$0 = 0 \longrightarrow \top$$

$$0 = S(x) \longrightarrow \bot$$

$$S(x) = 0 \longrightarrow \bot$$

$$S(x) = S(y) \longrightarrow x = y$$

$$(0/z)A \Rightarrow \forall x\ ((x/z)A \Rightarrow (S(x)/z)A) \Rightarrow \forall n\ (n/z)A$$

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0$$

$$S(x) \times y \longrightarrow x \times y + y$$

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A full example : arithmetic

$$0 = 0 \longrightarrow \top$$
$$0 = S(x) \longrightarrow \bot$$
$$S(x) = 0 \longrightarrow \bot$$
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A full example : arithmetic

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$$S(x) = 0 \longrightarrow \bot$$
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A full example : arithmetic

$$0 = 0 \longrightarrow \top$$
$$0 = S(x) \longrightarrow \bot$$
$$S(x) = 0 \longrightarrow \bot$$
$$S(x) = S(y) \longrightarrow x = y$$
$$\forall E\ (0 \in E \Rightarrow \forall x\ (x \in E \Rightarrow S(x) \in E) \Rightarrow \forall n\ n \in E))$$
$$\forall y_1 ... \forall y_n \forall z\ (z \in f_{z,y_1,...,y_n,A}(y_1, ..., y_n) \Leftrightarrow A)$$
$$0 + y \longrightarrow y$$
$$S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0$$
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A full example : arithmetic

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$$S(x) = 0 \longrightarrow \bot$$
$$S(x) = S(y) \longrightarrow x = y$$
$$\forall E\ (0 \in E \Rightarrow \forall x\ (x \in E \Rightarrow S(x) \in E) \Rightarrow \forall n\ (N(n) \Rightarrow n \in E)))$$
$$\forall y_1 ... \forall y_n \forall z\ (z \in f_{z,y_1,...,y_n,A}(y_1, ..., y_n) \Leftrightarrow A)$$
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A full example : arithmetic

$$0 = 0 \longrightarrow \top$$
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$$S(x) = 0 \longrightarrow \bot$$
$$S(x) = S(y) \longrightarrow x = y$$
$$N(n) \longrightarrow \forall E\,(0 \in E \Rightarrow \forall x\,(x \in E \Rightarrow S(x) \in E) \Rightarrow n \in E))$$
$$z \in f_{z,y_1,\dots,y_n,A}(y_1, \dots, y_n) \longrightarrow A$$
$$0 + y \longrightarrow y$$
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A full example : arithmetic

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INTERMISSION

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II. Models and cut elimination

A language $\mathcal{L}$

A structure $\mathcal{M}$ on $\mathcal{L}$ formed with

– a set M

– for each function symbol f of n arguments a function $\hat{f}$ from M^n to M

– for each predicate symbol P of n arguments a function $\hat{P}$ from M^n to $\{0, 1\}$

To each term is associated an element of M

To each formula is associated a truth value 0 or 1

Valid formula = associated to the value 1

Soundness : predicate logic

If for each formula A of Γ $[\![A]\!]^{\mathcal{M}} = 1$

and B is provable in Γ

then $[\![B]\!]^{\mathcal{M}} = 1$

Induction over proof structure

E.g.

$$\dfrac{\dfrac{\pi}{\Gamma \vdash P \wedge Q}}{\Gamma \vdash P} \wedge\text{-elim}$$

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Soundness : deduction modulo

If for each formula A of Γ $[\![A]\!]^{\mathcal{M}} = 1$

for all rules $l \longrightarrow r$ of $\mathcal{R}$ $[\![l]\!]^{\mathcal{M}} = [\![r]\!]^{\mathcal{M}}$

and B is provable in Γ $\mathcal{R}$

then $[\![B]\!]^{\mathcal{M}} = 1$

Induction over proof structure

E.g.

$$\frac{\dfrac{\pi}{\Gamma \vdash R}}{\Gamma \vdash P} \; R \equiv (P \wedge Q) \; \wedge\text{-elim}$$

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Consistency

If $[\![A]\!]^{\mathcal{M}} = 0$

then A is not provable from the axioms of Γ ($\Gamma\mathcal{R}$)

In particular if Γ ($\Gamma\mathcal{R}$) has a model

then $\perp$ not provable in Γ ($\Gamma\mathcal{R}$)

(and the converse : completeness)

Why restrict to $\{0, 1\}$?

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A bigger set so that instead of $0/1$

- $[\![A]\!]^{\mathcal{M}}$ is an intermediate truth value
- $[\![A]\!]^{\mathcal{M}}$ is a set of formulae that imply A
- $[\![A]\!]^{\mathcal{M}}$ is a set of sequents with conclusion A
- $[\![A]\!]^{\mathcal{M}}$ is a proof of A
- $[\![A]\!]^{\mathcal{M}}$ is a set of proofs of A
- we get completeness for weaker (*e.g.* constructive) logics

Examples

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Boolean algebras (*e.g.* $\mathcal{P}(\mathbb{N})$)

Heyting algebras

Truth value algebras

What is a boolean/Heyting algebra ?

An ordered set

with glb (for $\wedge$, $\forall$ and $\top$) and lub (for $\vee$, $\exists$ and $\bot$)

And a (weaker or stronger) complement (for $\Rightarrow$)

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Soundness : if for each formula A of Γ $[\![A]\!]^{\mathcal{M}} = Max$ and B is provable in Γ, then $[\![B]\!]^{\mathcal{M}} = Max$

$$\dfrac{\dfrac{\pi}{\Gamma \vdash P \wedge Q}}{\Gamma \vdash P} \wedge\text{-elim}$$

What are the key features ?

Order ? Not really

Soundness :

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Definition of $\tilde{\wedge}$, $\tilde{\Rightarrow}$, ...

$(a \mathbin{\tilde{\wedge}} b) \mathbin{\tilde{\Rightarrow}} a$ always in $\{Max\}$

... (closure by deduction rules)

Thus provable formulae are valid (induction over proof structure)

Truth values algebras

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$\langle \mathcal{B}, \mathcal{B}^+, \tilde{\wedge}, \tilde{\Rightarrow}, ... \rangle$

$(a \mathbin{\tilde{\wedge}} b) \mathbin{\tilde{\Rightarrow}} a$ always in $\mathcal{B}^+$, ...

Generalizes boolean/Heyting algebras (completeness for free)

Soundness : the closure conditions on $\mathcal{B}^+$ are (the weakest) sufficient conditions

An alternative presentation

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From a truth value algebra, we can define a relation

$$a \leq b \text{ iff } a \mathbin{\tilde{\Rightarrow}} b \in \mathcal{B}^+$$

Verifies all the properties of Heyting algebras except one : antisymmetry

A simple remark : antisymmetry is useless in the definition of Heyting algebras, it can be dropped

Only antisymmetry can : otherwise no closure by deduction rule

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Back to deduction modulo : a drawback of Heyting algebras

If $A \Leftrightarrow B$ valid then $[\![A]\!]^{\mathcal{M}} \leq [\![B]\!]^{\mathcal{M}}$ and $[\![B]\!]^{\mathcal{M}} \leq [\![A]\!]^{\mathcal{M}}$

In a boolean/Heyting algebra, due to antisymmetry
$[\![A]\!]^{\mathcal{M}} = [\![B]\!]^{\mathcal{M}}$

But not in a TVA

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If $A \Leftrightarrow B$ valid then $[\![A]\!]^{\mathcal{M}} \leq [\![B]\!]^{\mathcal{M}}$ and $[\![B]\!]^{\mathcal{M}} \leq [\![A]\!]^{\mathcal{M}}$

If $A \equiv B$ then $[\![A]\!]^{\mathcal{M}} = [\![B]\!]^{\mathcal{M}}$

A difference between logical and computational equivalence

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A non trivial example : $\{0, I, 1\}$

$\mathcal{B}^+ = \{I, 1\}$

$\tilde{\vee}$	0	I	1
0	0	1	1
I	1	1	1
1	1	1	1

same for $\tilde{\exists}$, $\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\forall}$, $\tilde{\bot}$

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But ...

$\tilde{\Rightarrow}$	0	I	1
0	1	1	1
I	0	1	1
1	0	I	I

A truth value algebra, not a Heyting algebra $I \leq 1$ and $1 \leq I$ but different

$P \longrightarrow (Q \Rightarrow Q)$

has a $\{0, 1\}$-valued model $[\![Q]\!] = 0$, $[\![P]\!] = 1$

and a $\{0, I, 1\}$-valued model (same)

$\tilde{\Rightarrow}$	0	I	1
0	1	1	1
I	0	1	1
1	0	I	I

Thm : has a $\mathcal{B}$-valued model for all $\mathcal{B}$

$P \longrightarrow (P \Rightarrow Q)$

has a $\{0, 1\}$-valued model $[\![P]\!] = [\![Q]\!] = 1$

but no $\{0, I, 1\}$-valued model

$\tilde{\Rightarrow}$	0	I	1
0	1	1	1
I	0	1	1
1	0	I	I

Super-consistency

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If $\mathcal{R}$ has a $\mathcal{B}$-valued model for some (non singleton) $\mathcal{B}$ then it is consistent

If $\mathcal{R}$ has a $\mathcal{B}$-valued model for all $\mathcal{B}$ then it is called super-consistent

Super-consistency

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$P \longrightarrow (Q \Rightarrow Q)$ super-consistent

$P \longrightarrow (Q \Rightarrow P)$ super-consistent

$P \longrightarrow (P \Rightarrow Q)$ not super-consistent (but consistent)

$P \longrightarrow (P \Rightarrow \exists x\ R(x))$ not super-consistent (but consistent)

Arithmetic, simple type theory, ... super-consistent

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The case of arithmetic

Give me a TVA $\mathcal{B}$

I build a model

The domains $\mathbb{N}$ and $\mathbb{N} \longrightarrow \mathcal{B}$

$\hat{0} = 0$, $\hat{S} = S$, $\hat{+} = +$, $\hat{\times} = \times$

$\hat{=}$ maps (n, p) to $\tilde{\top}$ if $n = p$ and to $\tilde{\bot}$ otherwise

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8/10 rules obviously valid

$$0 = 0 \longrightarrow \top$$
$$0 = S(x) \longrightarrow \bot$$
$$S(x) = 0 \longrightarrow \bot$$
$$S(x) = S(y) \longrightarrow x = y$$
$$N(n) \longrightarrow \forall E\ (0 \in E \Rightarrow \forall x\ (x \in E \Rightarrow S(x) \in E) \Rightarrow n \in E))$$
$$z \in f_{z,y_1,...,y_n,A}(y_1, ..., y_n) \longrightarrow A$$
$$0 + y \longrightarrow y$$
$$S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0$$
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$\hat{\in}$ is just application

Define $\hat{N}$ in such a way that

$N(n) \longrightarrow \forall E\ (0 \in E \Rightarrow \forall x\ (x \in E \Rightarrow S(x) \in E) \Rightarrow n \in E))$

is valid

and $\hat{f}_{z,y_1,...,y_n,A}$ in such a way that

$z \in f_{z,y_1,...,y_n,A}(y_1, ..., y_n) \longrightarrow A$

is valid

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A theorem

If a theory is super-consistent, then it has the cut elimination property

Two proofs :

(1) a normalization proof : we can eliminate cuts one by one and the process terminates (reducibility candidates are an example of (non Heyting) TVA)

(2) a simple cut elimination proof : if there is a proof there is a cut free proof

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The TVA of sequents

The elements of $\mathcal{S}$ are sets of sequents built with operations $\tilde{\wedge}$, $\tilde{\vee}$, ...

E.g. $a \mathbin{\tilde{\wedge}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv (A \wedge B)$ with $(\Gamma \vdash A) \in a$ and $(\Gamma \vdash B) \in b$

A non Heyting TVA : $(\tilde{\top} \mathbin{\tilde{\wedge}} \tilde{\top}) \neq \tilde{\top}$

A super-consistent has a $\mathcal{S}$-valued model $\mathcal{M}$

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$c \in \mathcal{S}$ contains only sequents that have a cut free proof

By induction on the construction of c

$c = (a \mathbin{\tilde{\wedge}} b)$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv (A \wedge B)$ with $(\Gamma \vdash A) \in a$ and $(\Gamma \vdash B) \in b$

The sequents $\Gamma \vdash A$ and $\Gamma \vdash B$ have cf proofs by IH, hence so does $\Gamma \vdash C$

If $\Gamma \vdash A$ is provable then $(\Gamma \vdash A) \in [\![A]\!]^{\mathcal{M}}$

If $\Gamma \vdash A$ is provable then $\Gamma \vdash A$ has a cut free proof