Objects as Constraints

A Formalism of Order-Sorted Featured Structures

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Motivation

- **Proposal:** a formalism for representing objects that is: intuitive (objects as labelled graphs), expressive ("real-life" data models), formal (logical semantics), operational (executable), and efficient (constraint-solving)
- **Why?** *viz.*, ubiquitous use of labelled graphs to structure information naturally as in:
  - object-orientation, knowledge representation,
  - databases, constraint-based programming,
  - natural language processing, graphical interfaces,
  - concurrency and communication,
  - XML, RDF, "Semantic Web," *etc.*, ...

Outline

- Motivation and background
- Basic order-sorted feature ($\mathcal{OSF}$) formalism
- Disjunction and negation
- Partial features, extensional sorts, relational features (aggregation)
- $\mathcal{OSF}$ theory unification
- Conclusion: A few fundamental principles...

Background

This work is the synthesis of research of many years by many people:

- Hassan Aït-Kaci (since 1983)
- Gert Smolka (since 1986)
- Andreas Podelski (since 1989)
- Franz Baader, Rolf Backhofen, Jochen Dörre, Martin Emele, Bernhard Nebel, Joachim Niehren, Ralf Treinen, Manfred Schmidt-Schauß, Remi Zajac, . . .

Basic $\mathcal{OSF}$ formalism

$\mathcal{OSF}$ **signature:**

$$\langle \mathcal{S}, \leq, \wedge, \mathcal{F} \rangle$$

s.t.:

- $\mathcal{S}$ is a set of sorts containing the sorts $\top$ and $\bot$
- $\leq$ is a partial order on $\mathcal{S}$ ($\bot$ is least element, $\top$ is greatest element)
- $\langle \mathcal{S}, \leq, \wedge \rangle$ is a lower semi-lattice ($s \wedge s'$ is called the greatest common subsort of s and s')
- $\mathcal{F}$ is a set of feature symbols.

$\mathcal{OSF}$ algebras

Given an $\mathcal{OSF}$ signature $\langle \mathcal{S}, \leq, \wedge, \mathcal{F} \rangle$ an $\mathcal{OSF}$ algebra is a structure:

$$\mathfrak{A} = \langle D^{\mathfrak{A}}, (s^{\mathfrak{A}})_{s \in \mathcal{S}}, (\ell^{\mathfrak{A}})_{\ell \in \mathcal{F}} \rangle$$

s.t.:

- $D^{\mathfrak{A}} \neq \emptyset$ is a set: the domain of $\mathfrak{A}$
- $s^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$ for s in $\mathcal{S}$ ($\top^{\mathfrak{A}} = D^{\mathfrak{A}}, \ \bot^{\mathfrak{A}} = \emptyset$)
- $(s \wedge s')^{\mathfrak{A}} = s^{\mathfrak{A}} \cap s'^{\mathfrak{A}}$
- $\ell^{\mathfrak{A}} : D^{\mathfrak{A}} \quad D^{\mathfrak{A}}$ for ℓ in $\mathcal{F}$ (*i.e.*, $\ell^{\mathfrak{A}}$ is a (total) function from the domain to the domain)

$\mathcal{OSF}$ homomorphism

$\mathcal{OSF}$ homomorphism between two $\mathcal{OSF}$ algebras $\mathfrak{A}$ and $\mathfrak{B}$:

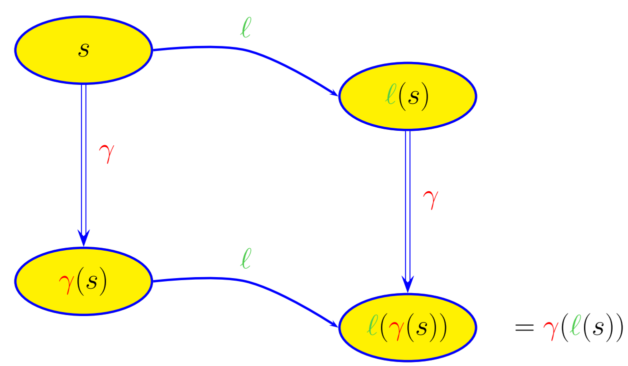
- $\gamma : D^{\mathfrak{A}} \quad D^{\mathfrak{B}}$
- $\gamma(\ell^{\mathfrak{A}}(d)) = \ell^{\mathfrak{B}}(\gamma(d))$ for all $d \in D^{\mathfrak{A}}$
- $\gamma(s^{\mathfrak{A}}) \subseteq s^{\mathfrak{B}}$ for all $s \in \mathcal{S}$

Taking $\mathfrak{A} = \mathfrak{B}$, γ is an $\mathcal{OSF}$ endomorphism of $\mathfrak{A}$:

- $\forall d \in D, \ \ \gamma(\ell(d)) = \ell(\gamma(d))$
- $\forall s \in \mathcal{S}, \ \ \gamma(s) \subseteq s$

This definition captures exactly inheritance of attributes.

Inheritance = $\mathcal{OSF}$ endomorphism



Hence, inheritance is endomorphic approximation.

$\mathcal{OSF}$ TERM SYNTAX

Let $\mathcal{V}$ be a countably infinite set of variables.

An $\mathcal{OSF}$ term is an expression of the form:

$$X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$$

where:

- $X \in \mathcal{V}$ is the root variable
- $s \in \mathcal{S}$ is the root sort
- $n \geq 0$ (if $n = 0$, we write $X : s$)
- $\{\ell_1, \ldots, \ell_n\} \subseteq \mathcal{F}$ are features
- $t_1, \ldots, t_n$ are $\mathcal{OSF}$ terms

EXAMPLE

$$\begin{aligned} X : person(&name \Rightarrow N : \top(first \Rightarrow F : string), \\ &name \Rightarrow M : id(last \Rightarrow S : string), \\ &spouse \Rightarrow P : person(name \Rightarrow I : id(last \Rightarrow S : \top), \\ &\qquad\qquad spouse \Rightarrow X : \top)). \end{aligned}$$

Lighter notation:

$$\begin{aligned} X : person(&name \Rightarrow \top(first \Rightarrow string), \\ &name \Rightarrow id(last \Rightarrow S : string), \\ &spouse \Rightarrow person(name \Rightarrow id(last \Rightarrow S), \\ &\qquad\qquad spouse \Rightarrow X)). \end{aligned}$$

$\mathcal{OSF}$ TERM SEMANTICS

- $\mathcal{OSF}$ term $t = X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$
- $\mathcal{OSF}$ interpretation $\mathfrak{A}$
- $\mathfrak{A}$-valuation $\alpha : \mathcal{V} \quad D^{\mathfrak{A}}$

Denotation of t in $\mathfrak{A}$ under valuation α:

$$[\![t]\!]^{\mathfrak{A},\alpha} \stackrel{\text{DEF}}{=} \{\alpha(X)\} \cap s^{\mathfrak{A}} \cap \left(\bigcap_{1 \leq i \leq n} (\ell_i^{\mathfrak{A}})^{-1}([\![t_i]\!]^{\mathfrak{A},\alpha}) \right)$$

Denotation of t in $\mathfrak{A}$ under all possible valuations:

$$[\![t]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} \bigcup_{\alpha : \mathcal{V} \quad D^{\mathfrak{A}}} [\![t]\!]^{\mathfrak{A},\alpha}.$$

$\mathcal{OSF}$ CLAUSE SYNTAX

For X and X' variables in $\mathcal{V}$, s a sort in $\mathcal{S}$, and ℓ a feature in $\mathcal{F}$, an $\mathcal{OSF}$ constraint is one of:

- $X : s$
- $X.\ell \doteq X'$
- $X \doteq X'$

An $\mathcal{OSF}$ clause is a conjunction of $\mathcal{OSF}$constraints—*i.e.*, a set of $\mathcal{OSF}$ constraints

- $\phi_1 \quad \& \quad \ldots \quad \& \quad \phi_n$

Semantics of $\mathcal{OSF}$ clauses

Satisfaction of $\mathcal{OSF}$ constraints in an $\mathcal{OSF}$ algebra $\mathfrak{A}$ by a valuation $\alpha : \mathcal{V} \quad D^{\mathfrak{A}}$ is defined by:

- $\mathfrak{A}, \alpha \models X : s$ iff $\alpha(X) \in s^{\mathfrak{A}}$
- $\mathfrak{A}, \alpha \models X \doteq Y$ iff $\alpha(X) = \alpha(Y)$
- $\mathfrak{A}, \alpha \models X.\ell \doteq Y$ iff $\ell^{\mathfrak{A}}(\alpha(X)) = \alpha(Y)$
- $\mathfrak{A}, \alpha \models \phi_1 \ \& \ \ldots \ \& \ \phi_n$ iff $\mathfrak{A}, \alpha \models \phi_i \quad \forall i = 1, \ldots, n$

From $\mathcal{OSF}$ terms to $\mathcal{OSF}$ clauses

An $\mathcal{OSF}$ term $t = X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$

is dissolved into an $\mathcal{OSF}$ clause $\phi(t)$ as follows:

$$\varphi(t) \stackrel{\text{DEF}}{=} X : s \ \& \ X.f_1 \doteq X_1 \ \& \ \ldots \ \& \ X.f_n \doteq X_n \ \& \ \varphi(t_1) \ \& \ \ldots \ \& \ \varphi(t_n)$$

where $X_1, \ldots, X_n$ are the root variables of $t_1, \ldots, t_n$.

Theorem: $\mathfrak{A}, \alpha \models \varphi(t) \iff [\![t]\!]^{\mathfrak{A},\alpha} \neq \emptyset$

Example of $\mathcal{OSF}$ term dissolution

$$\begin{aligned} t \ = \ X : person(&name \Rightarrow N : \top(first \Rightarrow F : string), \\ &name \Rightarrow M : id(last \Rightarrow S : string), \\ &spouse \Rightarrow P : person(name \Rightarrow I : id(last \Rightarrow S : \top), \\ &\qquad spouse \Rightarrow X : \top)) \end{aligned}$$

$$\begin{aligned} \varphi(t) = X : person \ & \& \ X\,.\,name \doteq N \ \& \ N : \top \\ & \& \ X\,.\,name \doteq M \ \& \ M : id \\ & \& \ X\,.\,spouse \doteq P \ \& \ P : person \\ & \& \ N\,.\,first \doteq F \ \& \ F : string \\ & \& \ M\,.\,last \doteq S \ \& \ S : string \\ & \& \ P\,.\,name \doteq I \ \& \ I : id \\ & \& \ I\,.\,last \doteq S \ \& \ S : \top \\ & \& \ P\,.\,spouse \doteq X \ \& \ X : \top \end{aligned}$$

Basic $\mathcal{OSF}$ term normalization

(1) Sort Intersection

$$\frac{\phi \ \& \ X : s \ \& \ X : s'}{\phi \ \& \ X : s \wedge s'}$$

(2) Inconsistent Sort

$$\frac{\phi \ \& \ X : \bot}{X : \bot}$$

(3) Variable Elimination

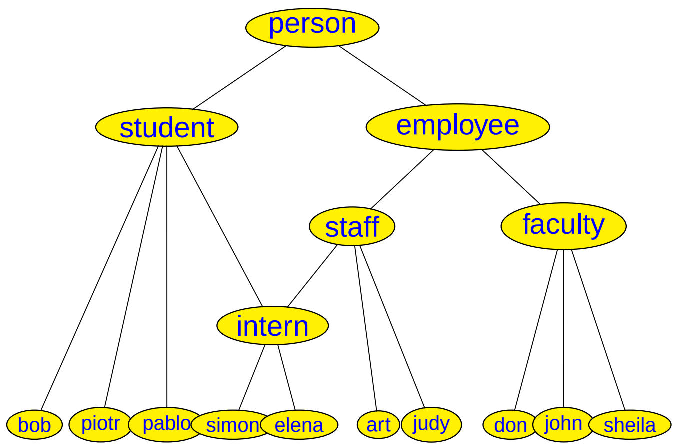
$$\frac{\phi \ \& \ X \doteq X'}{\phi[X'/X] \ \& \ X \doteq X'}$$

if $X \neq X'$
and $X \in \textbf{Var}(\phi)$

(4) Feature Functionality

$$\frac{\phi \ \& \ X.\ell \doteq X' \ \& \ X.\ell \doteq X''}{\phi \ \& \ X.\ell \doteq X' \ \& \ X' \doteq X''}$$

$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION



$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION

```
X : student
    (roommate => person(rep => E : employee),
     advisor  => don(secretary => E))

&

Y : employee
    (advisor  => don(assistant => A),
     roommate => S : student(rep => S),
     helper   => simon(spouse => A))

&

X = Y
```


$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION

```
X : intern
    (advisor => don(assistant => A,
                    secretary => S),
     helper => simon(spouse => A),
     roommate => S : intern(rep => S))

&

X = Y

&

E = S
```


EXTENDED $\mathcal{OSF}$ TERMS

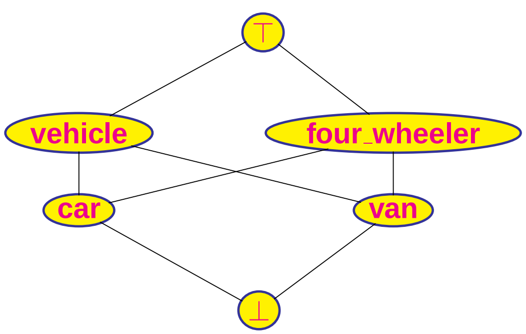
Basic $\mathcal{OSF}$ terms may be extended to express:

- Non-lattice sort signatures
- Disjunction
- Negation
- Partial features
- Extensional sorts (*i.e.*, denoting elements)
- Relational features (*a.k.a.*, "roles")
- Aggregates
- Sort definitions (*a.k.a.*, "$\mathcal{OSF}$ theories")

Extended $\mathcal{OSF}$ terms

```
OsfTerm ::= [ Variable: ] Term
   Term ::= ConjunctiveTerm
          | DisjunctiveTerm
          | NegativeTerm
ConjunctiveTerm ::= Sort [ ( Attribute+ ) ]
      Attribute ::= Feature ⇒ OsfTerm
DisjunctiveTerm ::= { OsfTerm [ ; OsfTerm ]* }
   NegativeTerm ::= ¬ OsfTerm
```


Enabling non-lattice signatures using disjunction



Non-unique GLBs are disjunctive sorts:

$$\textbf{vehicle} \wedge \textbf{four\_wheeler} \;=\; \{\textbf{car}; \textbf{van}\}$$

Disjunctive $\mathcal{OSF}$ terms

Syntax of disjunctive $\mathcal{OSF}$ terms:

$$\{\, t_1 \,;\, \ldots \,;\, t_n \,\}$$

Semantics of disjunctive $\mathcal{OSF}$ terms:

$$[\![\{t_1;\ldots;t_n\}]\!]^{\mathfrak{A},\alpha} \stackrel{\text{DEF}}{=} \bigcup_{1\leq i\leq n} [\![t_i]\!]^{\mathfrak{A},\alpha}$$

Disjunctive $\mathcal{OSF}$ clauses:

$$\varphi(\{t_1;\ldots;t_n\}) \stackrel{\text{DEF}}{=} \varphi(t_1) \parallel \ldots \parallel \varphi(t_n)$$

$\mathfrak{A},\alpha \models \phi_1 \parallel \ldots \parallel \phi_n$ iff $\mathfrak{A},\alpha \models \phi_i$ for some $i = 1,\ldots,n$

Disjunctive $\mathcal{OSF}$ normalization

(5) Non-unique GLB

$$\frac{\phi \;\&\; X:s \;\&\; X:s'}{\phi \;\&\; (X:s_1 \parallel \ldots \parallel X:s_n)}$$

where $\{s_i\}_{i=0}^{n} \stackrel{\text{DEF}}{=} \max_{\leq}\{t \in \mathcal{S} \mid t \leq s \text{ and } t \leq s'\}$

(6) Distributivity

$$\frac{\phi \;\&\; (\phi' \parallel \phi'')}{(\phi \;\&\; \phi') \parallel (\phi \;\&\; \phi'')}$$

(7) Disjunction

$$\frac{\phi \parallel \phi'}{\phi}$$

Negation

Syntax of negative $\mathcal{OSF}$ terms: $\neg t$

Semantics of negative $\mathcal{OSF}$ terms: $[\![\neg t]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} D^{\mathfrak{A}} \setminus [\![t]\!]^{\mathfrak{A}}$

Complemented sorts: $[\![\overline{s}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} D^{\mathfrak{A}} \setminus [\![s]\!]^{\mathfrak{A}}$

Sorted variable simplification:

$$\varsigma(X : s) \stackrel{\text{DEF}}{=} X : s \qquad \text{if } s \in \mathcal{S}$$

$$\varsigma(X : \overline{\overline{s}}) \stackrel{\text{DEF}}{=} \varsigma(X : s)$$

$$\varsigma(X : \overline{\{s_1; \ldots ; s_n\}}) \stackrel{\text{DEF}}{=} \varsigma(X : s_1) \ \& \ \ldots \ \& \ \varsigma(X : s_n)$$

Negative $\mathcal{OSF}$ terms

Dissolving negative $\mathcal{OSF}$ terms into $\mathcal{OSF}$ clauses eliminates negation:

$$\varphi(\neg(\neg t)) \stackrel{\text{DEF}}{=} \varphi(t)$$

$$\varphi(\neg\{t_1; \ldots ; t_n\}) \stackrel{\text{DEF}}{=} \varphi(\neg t_1) \ \& \ \ldots \ \& \ \varphi(\neg t_n)$$

$$\begin{aligned}\varphi(\neg X : s(\ell_i \Rightarrow t_i)_{i=1}^n) \stackrel{\text{DEF}}{=} \ & \varsigma(X : \overline{s}) \\ \| \ & X.\ell_1 \doteq X_1 \ \& \ \varphi(\neg t_1) \\ \| \ & X.\ell_1 \doteq X_1' \ \& \ X_1' \neq X_1 \ \& \ \varphi(t_1) \\ & \ldots \\ \| \ & X.\ell_n \doteq X_n \ \& \ \varphi(\neg t_n) \\ \| \ & X.\ell_n \doteq X_n' \ \& \ X_n' \neq X_n \ \& \ \varphi(t_n)\end{aligned}$$

Negative $\mathcal{OSF}$ term normalization

(8) Variable Disequality

$$\frac{\phi \ \& \ X \neq X}{\bot}$$

(9) Sort Complement

$$\frac{\phi \ \& \ X : \overline{s}}{\phi \ \& \ X : s'} \qquad \text{if } s' \in \max_{\leq}\{t \in \mathcal{S} \mid s \not\leq t \text{ and } t \not\leq s\}$$

Partial features

Partial features have restricted domains:

$$\exists y, \ \ y = \ell(x) \ \text{ only if } \ x \in \textbf{\textit{Dom}}(\ell)$$

Declaring partial feature domains:

$$\textbf{\textit{Dom}} : \mathcal{F} \quad \mathbf{2}^{\mathcal{S}}$$

s.t. $\textbf{\textit{Dom}}(\ell) \stackrel{\text{DEF}}{=}$ set of maximal sorts where ℓ is defined. Can also declare a feature's range: $\textbf{\textit{Ran}}_s : \mathcal{F} \quad \mathcal{S}$ for $s \in \textbf{\textit{Dom}}(\ell)$.

(10) Partial Feature

$$\frac{\phi \ \& \ X.\ell \doteq X'}{\phi \ \& \ X.\ell \doteq X' \ \& \ X : s \ \& \ X' : s'} \qquad \begin{array}{l}\text{if } \ s \in \textbf{\textit{Dom}}(\ell) \\ \text{and } \ \textbf{\textit{Ran}}_s(\ell) = s'\end{array}$$

Partial features (example)

Assume $\{nil, cons, list\} \subseteq \mathcal{S}$ such that:

$$\begin{array}{lcl} nil & < & list \\ cons & < & list \end{array}$$

and $\{hd, tl\} \subseteq \mathcal{F}$ such that:

$$\begin{array}{lcl} \textbf{\textit{Dom}}(hd) & \stackrel{\text{DEF}}{=} & \{cons\} \\ \textbf{\textit{Dom}}(tl) & \stackrel{\text{DEF}}{=} & \{cons\} \end{array}$$

then:

$$\begin{array}{lcl} list(tl \Rightarrow X) & \rightsquigarrow & cons(tl \Rightarrow X) \\ int(tl \Rightarrow X) & \rightsquigarrow & \perp \end{array}$$

Extensional sorts

The fact that some sorts denote singletons (*e.g.*, numbers) is not part of our axioms so far!

i.e.,

$$f(a \Rightarrow 1, b \Rightarrow 1) \not\leq f(a \Rightarrow X, b \Rightarrow X)$$

because:

$$f(a \Rightarrow X : s, b \Rightarrow X' : s) \leq f(a \Rightarrow Y, b \Rightarrow Y) \quad \text{iff} \quad X = X'$$

A sort that denotes a singleton, whenever all its images by a specific set of features do, is called extensional.

Extensional sorts

Extensional sorts are element constructors.

Let $\mathcal{E} \subseteq Minimals(\mathcal{S})$ be the set of extensional sorts with rank function:

$$\textbf{\textit{Arity}} : \mathcal{E} \qquad 2^{\mathcal{F}}$$

$$e.g.: \quad \begin{array}{lcl} \textbf{\textit{Arity}}(n) & = & \emptyset \quad \forall n \in \mathbb{N} \\ \textbf{\textit{Arity}}(nil) & = & \emptyset \\ \textbf{\textit{Arity}}(cons) & = & \{hd, tl\} \end{array}$$

Extensional sorts

Extensional sorts obey an axiom reminiscent of the axiom of functionality; *viz.*,

if $\textbf{\textit{Arity}}(f) = n$ and $X_i = Y_i \ (\forall i = 1, \ldots, n)$

then $f(X_1, \ldots, X_n) = f(Y_1, \ldots, Y_n)$

(11) Weak Extensionality

$$\frac{\phi \ \& \ X : s \ \& \ X' : s}{\phi \ \& \ X : s \ \& \ X \doteq X'}$$

if $s \in \mathcal{E}$ and $\forall \ell \in \textbf{\textit{Arity}}(s)$:

$\{X.f \doteq Y, X'.f \doteq Y\} \subseteq \phi$

Extensional sorts

The Weak Extensionality rule works, but not for cyclic terms; *viz.*:

let $s \in \mathcal{E}$ and $\textbf{\textit{Arity}}(s) = \{\ell\}$

then $X : s(\ell \Rightarrow X)$ & $X' : s(\ell \Rightarrow X')$

or $X : s(\ell \Rightarrow X')$ & $X' : s(\ell \Rightarrow X)$

are not reduced! So we need a stronger condition for cycles.

Strong extensionality

Proceed coinductively from roots to leaves carrying a context Γ, a set of pairs $s/\{X_1, \ldots, X_n\}$ s.t. $X_i \in \mathcal{V}$ $(i = 1, \ldots, n)$ and $s \in \mathcal{E}$ occurs at most once in Γ:

(12) Extensional Occurrence

$$\frac{\Gamma \uplus \{s/V, \ldots,\} \vdash \phi \;\&\; X : s}{\Gamma \uplus \{s/V \cup \{X\}, \ldots\} \vdash \phi \;\&\; X : s}$$

if $s \in \mathcal{E}$ and $X \notin V$ and $\forall f \in \textbf{\textit{Arity}}(s)$: $\{X.f \doteq X', X' : s'\} \subseteq \phi$ with $s' \in \mathcal{E}$

(13) Strong Extensionality

$$\frac{\Gamma \uplus \{s/\{X, X', \ldots\} \vdash \phi}{\Gamma \uplus \{s/\{X, \ldots\} \vdash \phi \;\&\; X \doteq X'}$$

if $s \in \mathcal{E}$

First-order terms as $\mathcal{OSF}$ terms

Let $\Sigma \stackrel{\text{DEF}}{=} \biguplus_{n \in \mathbb{N}} \Sigma_n$ be a ranked signature.

The first-order (rational) terms in $\mathcal{T}_{\Sigma,\mathcal{V}}$ are $\mathcal{OSF}$ terms s.t.:

- $\mathcal{S} \stackrel{\text{DEF}}{=} \Sigma \cup \{\top, \bot\}$ is a flat lattice
- $\mathcal{F} \stackrel{\text{DEF}}{=} \mathbb{N} \setminus \{0\}$
- $\textbf{\textit{Arity}}(\top) \stackrel{\text{DEF}}{=} \emptyset$
- $\textbf{\textit{Arity}}(\bot) \stackrel{\text{DEF}}{=} \{i \in \mathbb{N}^* \mid i \leq \max\{n > 0 \mid \Sigma_n \neq \emptyset\}\}$
- $\forall f \in \Sigma_n :$ $\textbf{\textit{Arity}}(f) \stackrel{\text{DEF}}{=} \{1, \ldots, n\}$
- $\forall i \in \mathcal{F} :$ $\textbf{\textit{Dom}}(i) \stackrel{\text{DEF}}{=} \bigcup_{i \leq n} \Sigma_n$
- $\forall i \in \mathcal{F}, \forall f \in \Sigma :$ $\textbf{\textit{Ran}}_f(i) \stackrel{\text{DEF}}{=} \begin{cases} \top & \text{if } f \in \textbf{\textit{Dom}}(i) \\ \bot & \text{otherwise} \end{cases}$

Relational features and aggregation

Relational features are set-valued features:

$$\forall \langle x, y \rangle \in A \times B : \quad \langle x, y \rangle \in R \;\; \text{iff} \;\; y \in R[x] \;\; \text{iff} \;\; x \in R^{-1}[y]$$

Sets are a particular case of monoidal aggregates:

- the notation "$X : s$" is generalized to carry an optional value $e \in \mathcal{E}$
- "$X = e : s$" means "X has value e of sort s" $(X \in \mathcal{V},\; e \in \mathcal{E},\; s \in \mathcal{S})$
- the shorthand "$X = e$" means "$X = e : \top$"
- when the sort $s \in \mathcal{S}$ denotes a commutative monoid $\langle \star, \mathbf{1}_\star \rangle$, the shorthand "$X : s$" means "$X = \mathbf{1}_\star : s$."

Relational features and aggregation

The semantic conditions are thus extended:

$$\mathfrak{A}, \alpha \models X = e : s \text{ iff } e^{\mathfrak{A}} \in s^{\mathfrak{A}} \text{ and } \alpha(X) = e^{\mathfrak{A}}$$

(14) Value Aggregation

$$\frac{\phi \;\&\; X = e : s \;\&\; X = e' : s'}{\phi \;\&\; X = e \star e' : s \wedge s'}$$

if s and s' are both subsorts of commutative monoid $\langle \star, \mathbf{1}_\star \rangle$

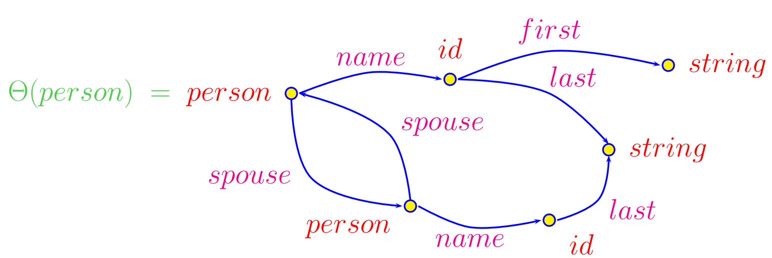
N.B.: This works for any commutative monoid—not just sets!

$\mathcal{OSF}$ theory unification

IDEA: Augment the sort ordering with constraints imposing:

- sorts of features
- coreference equations

e.g., define the sort person to abide by the structure:



$\mathcal{OSF}$ theory

An $\mathcal{OSF}$ theory is a function: $\Theta : \mathcal{S} \quad \Psi$

An $\mathcal{OSF}$ theory is order-consistent iff it is monotonic:

$$s \leq s' \;\Rightarrow\; \Theta(s) \leq \Theta(s')$$

$\mathcal{OSF}$ theory unification problem:

> Given an order-consistent $\mathcal{OSF}$ theory Θ, normalize any term of sort s taking into account the $\mathcal{OSF}$ constraints $\Theta(s)$.

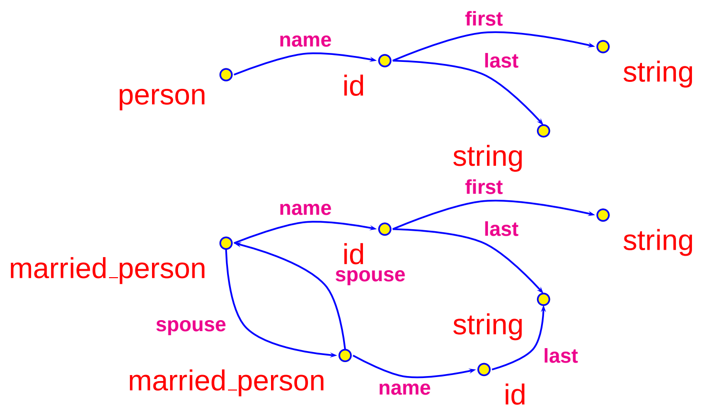
Theorem $\mathcal{OSF}$ theory unification is undecidable.

$\mathcal{OSF}$ theory

However... there is an algorithm such that:

- inconsistent terms are always normalized to $\perp$ in finitely many steps;
- normalization can perform $\mathcal{OSF}$ constraint inheritance from the theory lazily;
- there is an efficient algorithm which is complete for a large class of $\mathcal{OSF}$ theories;
- only one rule completes it (and may cause divergence).

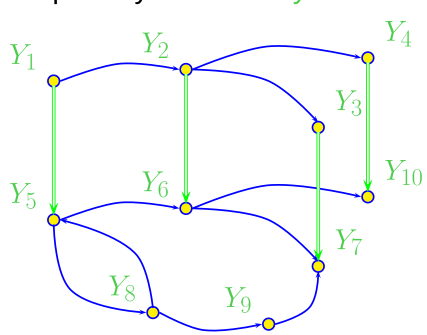
$\mathcal{OSF}$ THEORY UNIFICATION (EXAMPLE)



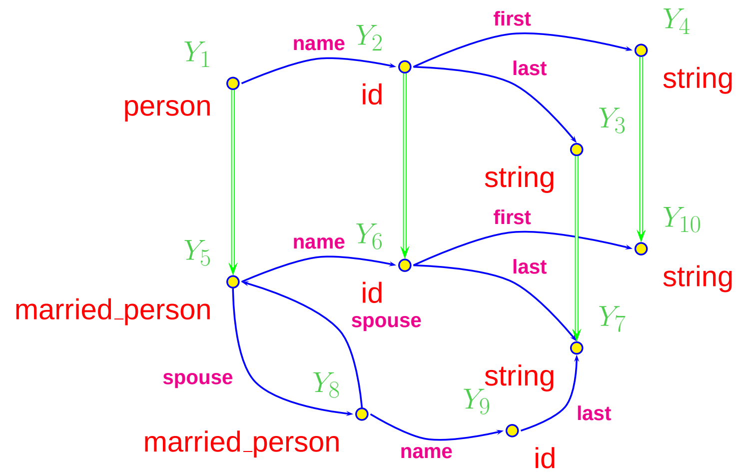
$\mathcal{OSF}$ THEORY UNIFICATION

The fact that an $\mathcal{OSF}$ theory is order-consistent yields an endomorphic mapping of theory variables.

In particular, the sort ordering $\leq$ and the GLB operation $\wedge$ extend homomorphically to all theory variables.



$\mathcal{OSF}$ THEORY UNIFICATION



$\mathcal{OSF}$ THEORY UNIFICATION

Normalizing:

$$
\begin{array}{ll}
 & P : person(name \Rightarrow \top(last \Rightarrow \text{“Smith”})) \\
\& & P : married\_person(spouse \Rightarrow Q) \\
\& & Q : person(name \Rightarrow id(last \Rightarrow S))
\end{array}
$$

yields, among other things:

$$
\begin{array}{ll}
 & P : married\_person \\
\& & Q : married\_person \\
\& & S : \text{“Smith”} \ldots
\end{array}
$$

$\mathcal{OSF}$ THEORY UNIFICATION

(0) Frame Allocation

$$\frac{\Gamma \qquad\qquad \vdash \; X : s \;\; \& \;\; \phi}{\Gamma \bigcup \{\{X \backslash Y_s\}\} \;\vdash\; X : s \;\; \& \;\; \phi}$$

if $\forall s' \in \mathcal{S},\ \forall F \in \Gamma :\ X \backslash Y_{s'} \notin F$

$\mathcal{OSF}$ THEORY UNIFICATION (EMPTY THEORY)

(1) Sort Intersection

$$\frac{\Gamma \bigcup \{\{X \backslash Y_{s'}\} \cup F\} \;\vdash\; X : s \;\; \& \;\; X : s' \;\; \& \;\; \phi}{\Gamma \bigcup \{\{X \backslash Y_{s \wedge s'}\} \cup F\} \;\vdash\; X : s \wedge s' \;\; \& \;\; \phi}$$

(2) Inconsistent Sort

$$\frac{\Gamma \bigcup \{\{X \backslash Y_{\perp}\} \cup F\} \;\vdash\; \phi}{\emptyset \qquad\qquad \vdash\; \perp}$$

$\mathcal{OSF}$ THEORY UNIFICATION (EMPTY THEORY)

(3) Variable Elimination

$$\frac{\Gamma \qquad \vdash\; X \doteq X' \;\; \& \;\; \phi}{\Gamma[X'/X] \;\vdash\; X \doteq X' \;\; \& \;\; \phi[X'/X]}$$

if $\quad X \neq X'$
and $\ X \in \mathbf{\mathit{Var}}(\Gamma) \cup \mathbf{\mathit{Var}}(\phi)$

(4) Feature Functionality

$$\frac{\Gamma \;\vdash\; X.\ell \doteq X' \;\; \& \;\; X.\ell \doteq X'' \;\; \& \;\; \phi}{\Gamma \;\vdash\; X.\ell \doteq X' \;\; \& \;\; X' \doteq X'' \;\; \& \;\; \phi}$$

$\mathcal{OSF}$ THEORY UNIFICATION (NON-EMPTY THEORY)

(5) Feature Inheritance (if $\ell(Y) = Y'$ and $X' \backslash Y' \notin F$)

$$\frac{\Gamma \bigcup \{\{X \backslash Y\} \cup F\} \qquad \vdash\; \phi \;\; \& \;\; X.\ell \doteq X'}{\Gamma \bigcup \{\{X \backslash Y, X' \backslash Y'\} \cup F\} \;\vdash\; \phi \;\; \& \;\; X.\ell \doteq X' \;\; \& \;\; X' : \mathbf{\mathit{Sort}}(Y')}$$

(6) Frame Merging

$$\frac{\Gamma \bigcup \{\{X \backslash Y_s\} \cup F, \{X \backslash Y_{s'}\} \cup F'\} \;\vdash\; \phi}{\Gamma \bigcup \{\{X \backslash Y_{s \wedge s'}\} \cup F \cup F'\} \qquad \vdash\; \phi}$$

$\mathcal{OSF}$ THEORY UNIFICATION (NON-EMPTY THEORY)

(7) Frame Reduction

$$\frac{\Gamma \bigcup \{\{X \backslash Y, X \backslash Y'\} \cup F\} \ \vdash \ \phi}{\Gamma \bigcup \{\{X \backslash Y\} \cup F\} \ \vdash \ \phi} \qquad \text{if } Y \leq Y'$$

(8) Theory Coreference

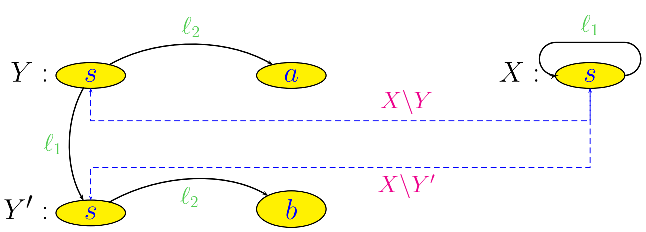
$$\frac{\Gamma \bigcup \{\{X \backslash Y, X' \backslash Y\} \cup F\} \ \vdash \ \phi}{\Gamma \bigcup \{\{X \backslash Y\} \cup F\} \ \vdash \ \phi \ \& \ X \doteq X'}$$

$\mathcal{OSF}$ THEORY UNIFICATION (STRONG NORMALIZATION)

(9) Theory Feature Completion

$$\frac{\Gamma \ \vdash \ \phi}{\Gamma \ \vdash \ X.\ell \doteq Z \ \& \ \phi}$$

if $X \backslash Y \in F$ for some $F \in \Gamma$
and $X \backslash Y' \in F'$ for some $F' \in \Gamma$
and both $\ell(Y)$, $\ell(Y')$ exist
and Z is new



CONCLUSION

We have overviewed a formalism of objects where:

- "real-life" objects are viewed as logical constraints
- objects may be approximated as set-denoting constructs
- object normalization rules provide an efficient operational semantics
- consistency extends unification (and thus matching)
- this enables rule-based computation (whether rewrite or logical rules) over general graph-based objects
- this yield a powerful means for effectively using ontologies

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Thank You For Your Attention !